Engineering Example A1:
Beam with Point Load

This worksheet derives the shear, moment, slope, and deflection of a beam from a point load located anywhere along the length of the beam.

It is more complex than Engineering Example 13.1 because it requires the use of step functions.

Field: Civil Engineering

Features used:
Integration
User-defined functions
Boolean logic
Symbolics
Solving
Solve blocks
Vectorize
Plots

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In Engineering Example 13.1, we derived the equations used to plot the shear, moment, slope, and deflection of a uniformly loaded beam. In this example we derive the shear, moment, slope, and deflection of a beam with a point load.

This example is a little more complex because the function is a step function and requires the use of the ***if*** function.

Note: PTC Mathcad does not include the integration constant C. Because we are creating functions for multiple solutions, we will need to solve for the integration constant.

In this example, all variables were manually assigned the "Variable" label.

Use the following values to check the derived formulas for numeric results.

$\mathrm{Length} := 20 \cdot ft$ $\quad P := 1000 \cdot lbf$ $\quad E_1 := 29000 \cdot ksi$ $\quad I_1 := 428 \cdot in^4$

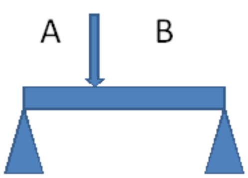
$A := 7\ ft$

$B := \mathrm{Length} - A = 13.00\ ft$

$$z := 0\ ft, 0.1\ ft .. \mathrm{Length} = \begin{bmatrix} 0.00 \\ 0.10 \\ 0.20 \\ 0.30 \\ 0.40 \\ 0.50 \\ \vdots \end{bmatrix} ft$$

Load

Since load is downward, use a negative value.



$$R_{PointLeft}(p, a, b) := \frac{p \cdot b}{a + b}$$

$$R_{PointLeft}(P, A, B) = 650.00\ lbf$$

$$R_{PointRight}(p, a, b) := \frac{p \cdot a}{a + b}$$

$$R_{PointRight}(P, A, B) = 350.00\ lbf$$

Given a point load p located anywhere along the length of a beam (at distance a), derive the fromulas for shear, moment, slope, and deflection anywhere along the length of a beam.

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Calculate Shear

The shear to the left of the point load is the left reaction. The shear to the right of the point load is the right reaction. Create a function that will determine the shear based on whether the location is left or right of the point load.

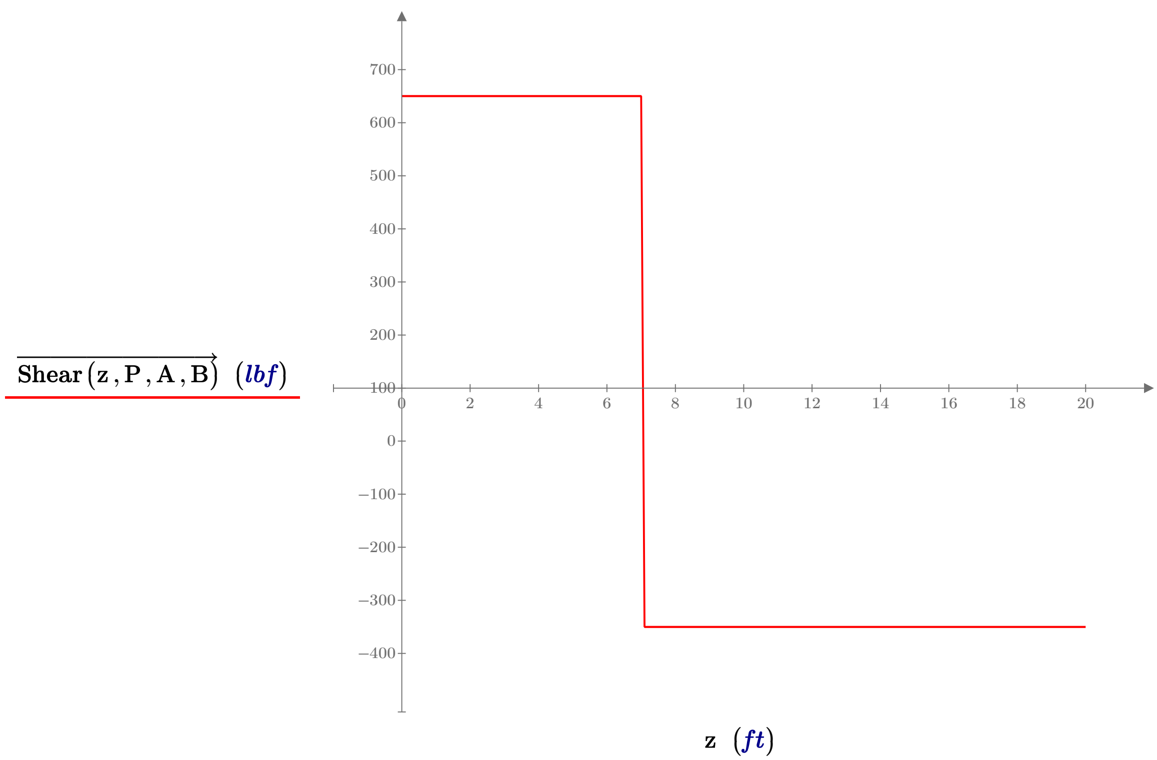
$V1(p, a, b) := R_{PointLeft}(p, a, b)$ $\qquad$ $V1(P, A, B) = 650.00\ lbf$

$V2(p, a, b) := R_{PointLeft}(p, a, b) - p$ $\qquad$ $V2(P, A, B) = -350.00\ lbf$

$Shear(x, p, a, b) := if(x \le a, V1(p, a, b), V2(p, a, b))$

Plot the Shear function.

The function Shear expects a value at a single point. In order to calculate solutions at many points, use the ***vectorize*** operator.



Calculate Moment

Calculate the moment by integrating the area under the shear diagram. The shear diagram is a step function, so we need functions to calculate the area to the left of the point load and to the right of the point load.

Since we do not know the value of the constant of integration C, include it as a variable to the function.

$$\mathrm{M1}(x,p,a,b,C2a) := \int \mathrm{V1}(p,a,b)\,dx + C2a \rightarrow C2a + \frac{b \cdot p \cdot x}{a+b}$$

$$\mathrm{M2}(x,p,a,b,C2a,C2b) := \mathrm{M1}(a,p,a,b,C2a) + \int \mathrm{V2}(p,a,b)\,dx + C2b \xrightarrow{simplify} \frac{a \cdot C2a + a \cdot C2b + b \cdot C2a + b \cdot C2b + a \cdot b \cdot p - a \cdot p \cdot x}{a+b}$$

$$\mathrm{M2}(x,p,a,b,C2a,C2b) \rightarrow \frac{a \cdot C2a + a \cdot C2b + b \cdot C2a + b \cdot C2b + a \cdot b \cdot p - a \cdot p \cdot x}{a+b}$$

Mathcad Prime 3.0 does not allow for symbolics in solve blocks, so use symbolics and the **solve** keyword.

$$\begin{bmatrix} C2a & C2b \end{bmatrix} := \begin{bmatrix} 0 = \mathrm{M1}(0,p,a,b,C2a) \\ 0 = \mathrm{M2}(b,p,a,b,C2a,C2b) \end{bmatrix} \xrightarrow{solve, C2a, C2b} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$C2a \rightarrow 0$ $\quad$ $C2b \rightarrow 0$

Now that we know C2a and C2b, we can eliminate them as variables

$$\mathrm{M1}(x,p,a,b) := \mathrm{M1}(x,p,a,b,C2a) \rightarrow \frac{b \cdot p \cdot x}{a+b}$$

$$\mathrm{M2}(x,p,a,b) := \mathrm{M2}(x,p,a,b,C2a,C2b) \xrightarrow{simplify} \frac{a \cdot p \cdot (b-x)}{a+b}$$

$\mathrm{M1}(A,P,A,B) = 4550.00\ ft \cdot lbf$ Moment at location of point load

$\mathrm{M2}(0,P,A,B) = 4550.00\ ft \cdot lbf$ Moment at location of point load. Remember that M2 starts at the location of the point load.

$$\mathrm{Moment}(x,p,a,b) := \mathrm{if}(x \le a, \mathrm{M1}(x,p,a,b), \mathrm{M2}(x-a,p,a,b))$$

The x-a in M2 is because M2 starts at the location of the point load.

Create a formula for maximum moment. This occurs at the location of the point load, at "a."

$$\mathrm{MaxMoment}(p,a,b) := \mathrm{Moment}(a,p,a,b) \rightarrow \frac{a \cdot b \cdot p}{a+b}$$

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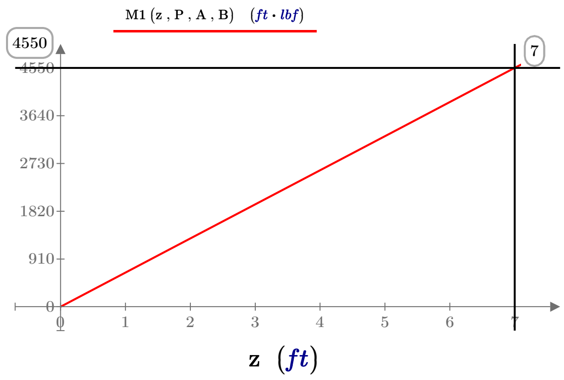
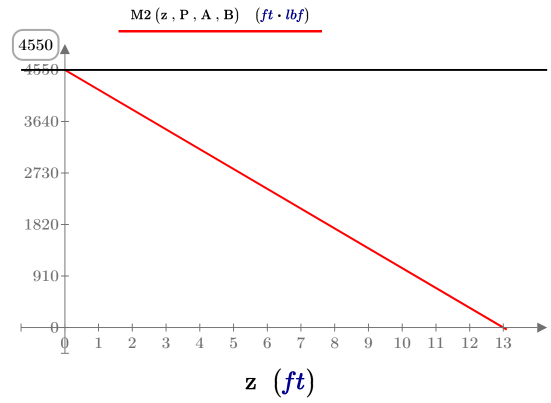
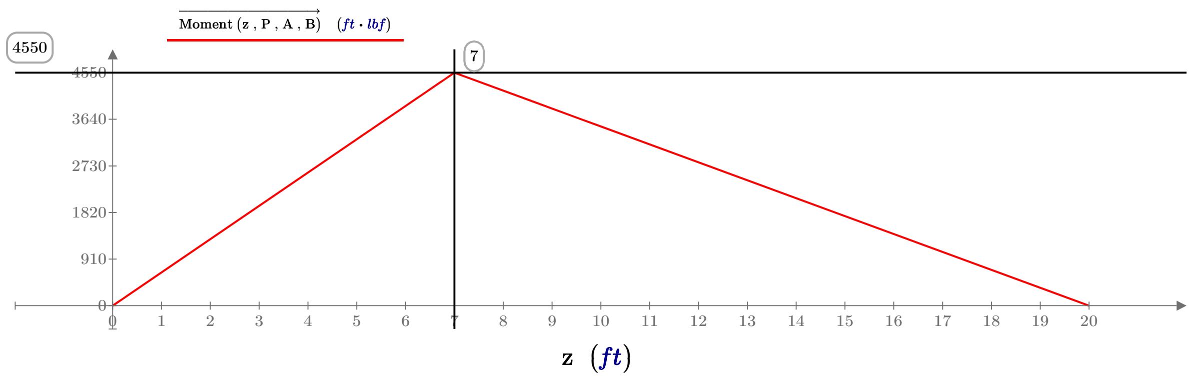


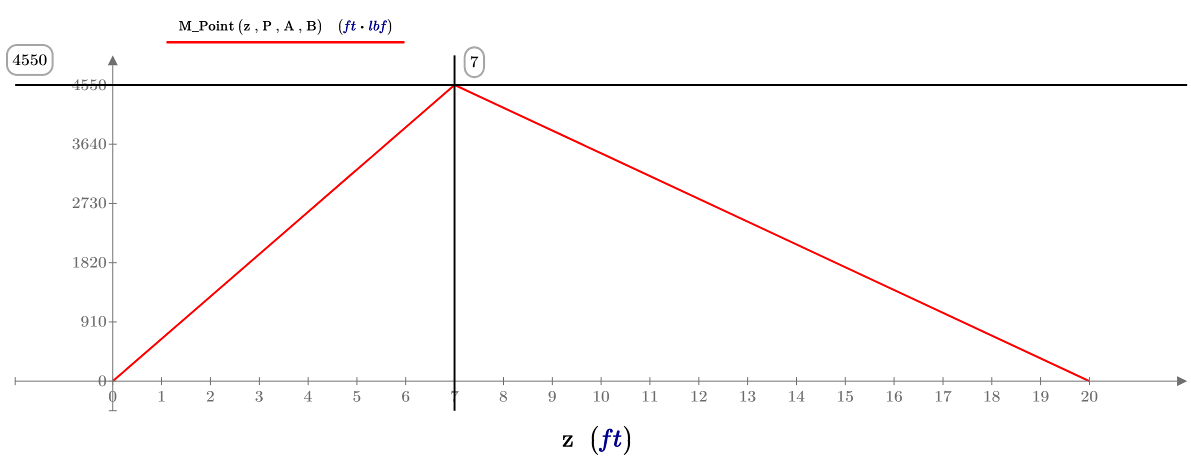




Verify the moment function with a different method of calculation. This method takes a free-body diagram at x and calculates the moment.

$Maximum(x, y) := max(x, y)$ The user-defined function ***Maximum*** allows the use of the ***vectorize*** operator on the ***max*** function.

$$M\_Point(x, p, a, b) := R_{PointLeft}(p, a, b) \cdot x - \overrightarrow{P \cdot Maximum(0, x - a)}$$



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Calculate Slope

From mechanics of materials, the relationship for the radius of curvature of a beam p and moment is defined as $\frac{1}{p} = \frac{M}{E \cdot I}$, and the relationship of moment to slope θ is defined as $M = E \cdot I \cdot \frac{d}{dx}\theta$.
Calculate slope θ by integrating M/EI.

$$\mathrm{Slope1}(x,p,a,b,E,I,C3a) := \int \frac{M1(x,p,a,b)}{E \cdot I}\,dx + C3a \rightarrow C3a + \frac{b \cdot p \cdot x^2}{2 \cdot E \cdot I \cdot (a+b)}$$

$$\mathrm{Slope2}(x,p,a,b,E,I,C3b) := \int \frac{M2(x,p,a,b)}{E \cdot I}\,dx + C3b \rightarrow C3b - \frac{a \cdot p \cdot (b-x)^2}{2 \cdot E \cdot I \cdot (a+b)}$$

Solve for the constants of integration after deflections are calculated.

Calculate Deflection

$$\Delta 1(x,p,a,b,E,I,C3a,C4a) := \int \mathrm{Slope1}(x,p,a,b,E,I,C3a)\,dx + C4a \longrightarrow C4a + x \cdot C3a + \frac{b \cdot p \cdot x^3}{6 \cdot E \cdot I \cdot (a+b)}$$

The deflection at x=0 is zero. Use this to solve for C4a.

$$C4a := 0 = \Delta 1(0,p,a,b,E,I,C3a,C4a) \xrightarrow{solve,\,C4a} 0$$
$$C4a \rightarrow 0$$

$$\Delta 2(x,p,a,b,E,I,C3b,C4b) := \int \mathrm{Slope2}(x,p,a,b,E,I,C3b)\,dx + C4b \xrightarrow{simplify} \frac{3 \cdot a \cdot p \cdot b \cdot x^2 - 3 \cdot a \cdot p \cdot b^2 \cdot x + 6 \cdot E \cdot I \cdot C3b}{}$$

$$\Delta 2(x,p,a,b,E,I,C3b,C4b) \rightarrow \frac{3 \cdot a \cdot p \cdot b \cdot x^2 - 3 \cdot a \cdot p \cdot b^2 \cdot x + 6 \cdot E \cdot I \cdot C3b \cdot b \cdot x + 6 \cdot E \cdot I \cdot C4b \cdot b - a \cdot p \cdot x^3 + 6 \cdot E \cdot I \cdot a \cdot C3b \cdot x}{6 \cdot E \cdot I \cdot (a+b)}$$

The deflection at x=0 is zero. Use this to solve for C4b. Remember that this function begins at the point load, and the deflection to the right will begin to increase.

$$C4b := 0 = \Delta 2(0,p,a,b,E,I,C3b,C4b) \xrightarrow{solve,\,C4b} 0$$
$$C4b \rightarrow 0$$

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Solve for C3a and C3b. The constraints: 1) The sum of the deflection 1 at x=a and of the deflection 2 at x=b is zero, and 2) the slopes are equal where Slope1 and Slope2 connect.

$$[C3a\ \ C3b] := \begin{bmatrix} 0 = \Delta1(a,p,a,b,E,I,C3a,C4a) + \Delta2(b,p,a,b,E,I,C3b,C4b) \\ Slope1(a,p,a,b,E,I,C3a) = Slope2(0,p,a,b,E,I,C3b) \end{bmatrix} \xrightarrow[simplify]{solve,\,C3a,\,C3b} \left[-\frac{a \cdot b \cdot p \cdot (a+2 \cdot b)}{6 \cdot E \cdot I \cdot a + 6 \cdot E \cdot I \cdot b} \right.$$

$$C3a \xrightarrow{simplify} -\frac{a \cdot b \cdot p \cdot (a+2 \cdot b)}{6 \cdot E \cdot I \cdot a + 6 \cdot E \cdot I \cdot b}$$

$$C3b \xrightarrow{simplify} \frac{a \cdot b \cdot p \cdot (2 \cdot a+b)}{6 \cdot (E \cdot I \cdot a + E \cdot I \cdot b)}$$

Now that we know the constants of integration C3a, C3b, C4a, and C4b, we can eliminate them from all functions.

$$Slope1(x,p,a,b,E,I) := Slope1(x,p,a,b,E,I,C3a) \xrightarrow{simplify} -\frac{b \cdot p \cdot (a^2 + 2 \cdot b \cdot a - 3 \cdot x^2)}{6 \cdot E \cdot I \cdot (a+b)}$$

$$Slope2(x,p,a,b,E,I) := Slope2(x,p,a,b,E,I,C3b) \xrightarrow{simplify} -\frac{a \cdot p \cdot (2 \cdot b^2 - 6 \cdot b \cdot x - 2 \cdot a \cdot b + 3 \cdot x^2)}{6 \cdot E \cdot I \cdot (a+b)}$$

$$Slope(x,p,a,b,E,I) := \text{if}(x \le a, Slope1(x,p,a,b,E,I), Slope2(x-a,p,a,b,E,I))$$

$$\Delta1(x,p,a,b,E,I) := \Delta1(x,p,a,b,E,I,C3a,C4a) \xrightarrow{simplify} -\frac{b \cdot p \cdot x \cdot (a^2 + 2 \cdot b \cdot a - x^2)}{6 \cdot E \cdot I \cdot (a+b)}$$

$$\Delta2(x,p,a,b,E,I) := \Delta2(x,p,a,b,E,I,C3b,C4b) \xrightarrow{simplify} -\frac{a \cdot p \cdot x \cdot (2 \cdot b^2 - 3 \cdot b \cdot x - 2 \cdot a \cdot b + x^2)}{6 \cdot E \cdot I \cdot (a+b)}$$

$$\Delta3(x,p,a,b,E,I) := \text{if}(x \le a, \Delta1(x,p,a,b,E,I), \Delta1(a,p,a,b,E,I) + \Delta2(x-a,p,a,b,E,I))$$

Engineering Example A1
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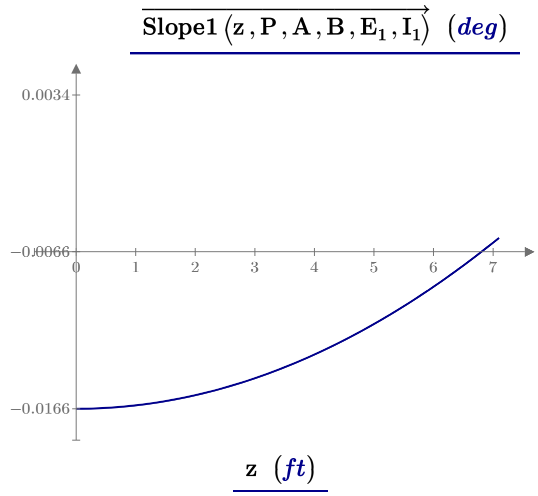
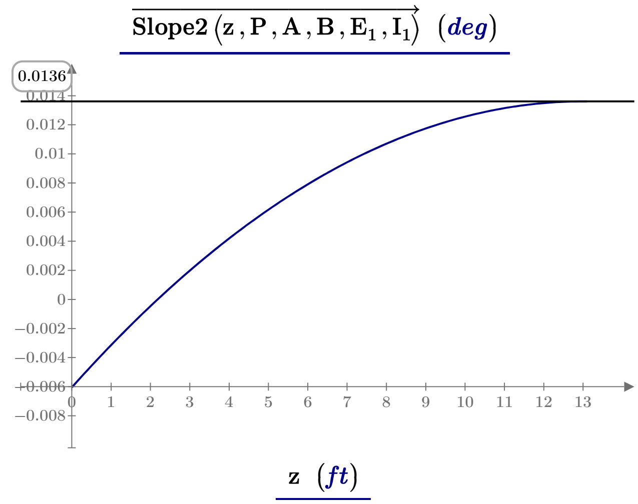




$\mathbf{Slope1}\left(\mathrm{A},\mathrm{P},\mathrm{A},\mathrm{B},\mathrm{E_1},\mathrm{I_1}\right)=-0.006\ \boldsymbol{deg}$ $\mathbf{Slope2}\left(0\ \boldsymbol{ft},\mathrm{P},\mathrm{A},\mathrm{B},\mathrm{E_1},\mathrm{I_1}\right)=-0.006\ \boldsymbol{deg}$

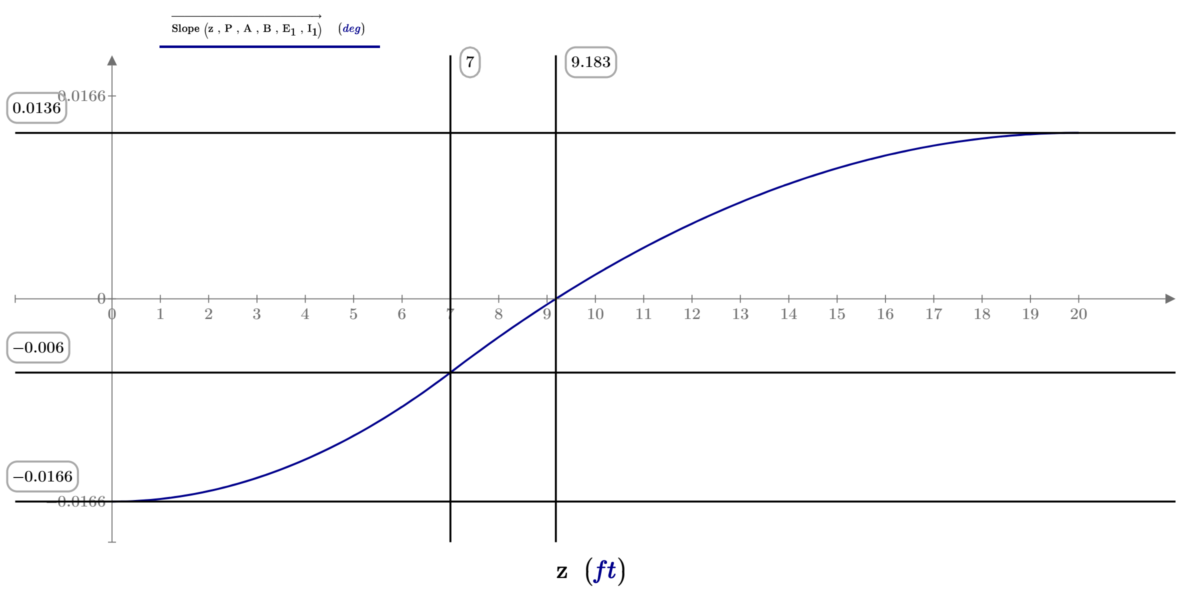
Find location where slope is zero using a solve block. This will be the point of maximum deflection.

$\mathrm{D}:=\mathrm{A}$

$0=\mathit{Slope}\left(\mathrm{D},\mathrm{P},\mathrm{A},\mathrm{B},\mathrm{E_1},\mathrm{I_1}\right)$

$\mathrm{Distance}:=\mathrm{Find}\left(D\right)=9.183\ \boldsymbol{ft}$

$\mathrm{Distance}=9.183\ \boldsymbol{ft}$



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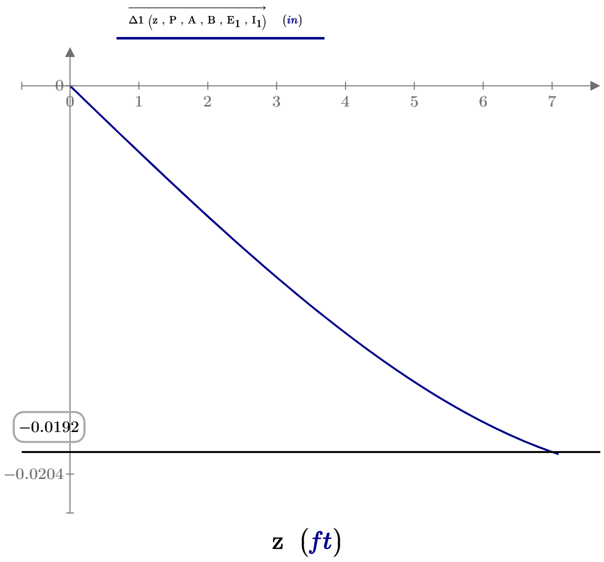
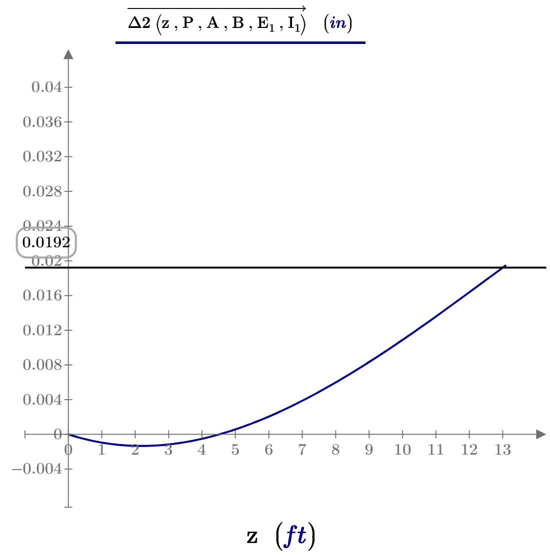
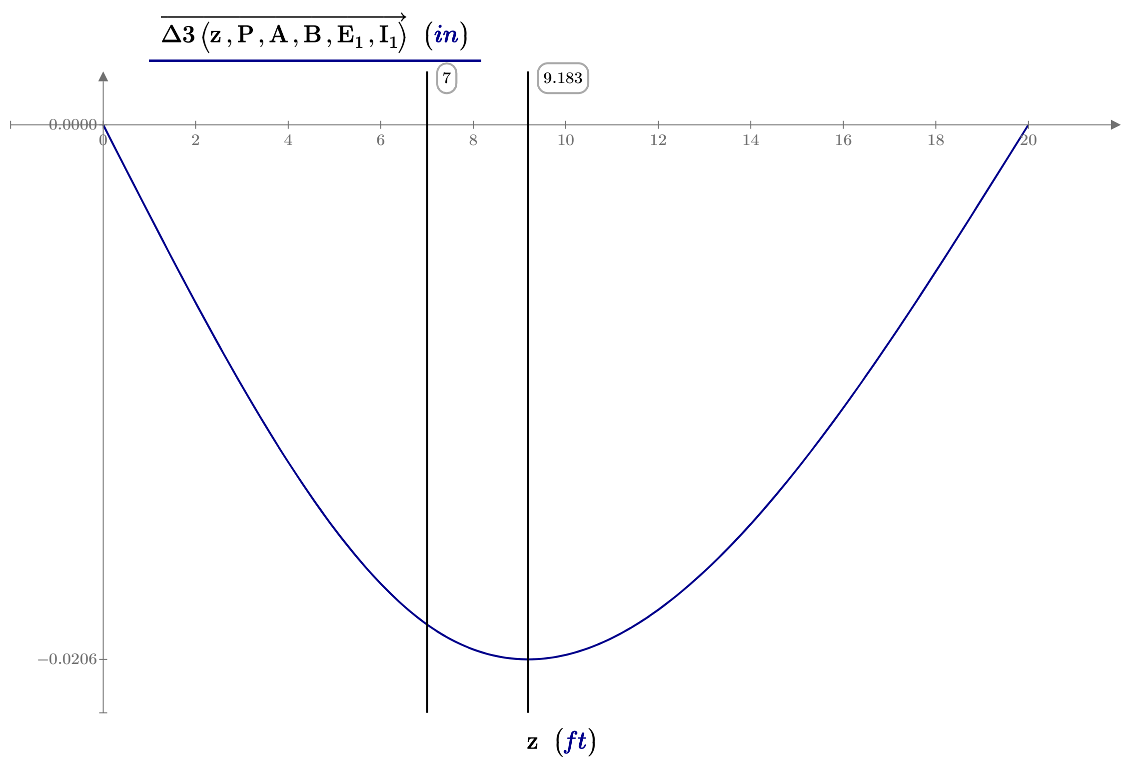




Find maximum deflection at the point of zero slope (Distance)

$$\mathbf{MaxDeflection} := \mathbf{\Delta 3}\left(\text{Distance}, \mathbf{P}, \mathbf{A}, \mathbf{B}, \mathbf{E_1}, \mathbf{I_1}\right) = -0.0206\ in$$



Engineering Example A1
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Now, let's derive the general formula for maximum moment. The maximum moment occurs where the slope is zero. The point of zero slope is dependant upon whether or not a<b, so we must derive two formulas, and then use a conditional expression to select the appropriate formula.

Find location where Slope1 is zero. Use the **fully** modifier, which will provide the solution along with the constraints.

$$\mathbf{Slope1}(x,p,a,b,E,I) \xrightarrow[\textit{fully}]{\textit{solve},x} \begin{cases} \begin{bmatrix} \dfrac{\sqrt{3}\cdot\sqrt{a}\cdot\sqrt{a+2\cdot b}}{3} \\ -\dfrac{\sqrt{3}\cdot\sqrt{a}\cdot\sqrt{a+2\cdot b}}{3} \end{bmatrix} & \text{if } \_c1\in\mathbb{C}\wedge(a+b=0\wedge b\neq0\wedge p\neq0\vee a+ \ \ldots \\ \textit{undefined} & \text{else} \end{cases}$$

Find location where Slope2 is zero.

$$\mathbf{Slope2}(x,p,a,b,E,I) \xrightarrow[\textit{fully}]{\textit{solve},x} \begin{cases} \begin{bmatrix} b+\dfrac{\sqrt{3}\cdot\sqrt{b\cdot(2\cdot a+b)}}{3} \\ b-\dfrac{\sqrt{3}\cdot\sqrt{b\cdot(2\cdot a+b)}}{3} \end{bmatrix} & \text{if } \_c1\in\mathbb{C}\wedge(a+b=0\wedge a\neq0\wedge p\neq0\vee a+b\neq0\wedge a\neq0\wedge p\neq0\wedge I=0\vee a+b\neq0\wedge I\neq0\wedge a\neq0\wedge p\neq0\wedge E \\ \textit{undefined} & \text{else} \end{cases}$$

$$\mathbf{\Delta AtZeroSlope1}(\mathbf{p},\mathbf{a},\mathbf{b},\mathbf{E},\mathbf{I}) := \Delta 1\left(\frac{\sqrt{3}\cdot\sqrt{a}\cdot\sqrt{a+2\cdot b}}{3},p,a,b,E,I\right) \xrightarrow{\textit{simplify}} -\frac{\sqrt{3}\cdot a^{\frac{3}{2}}\cdot b\cdot p\cdot(a+2\cdot b)^{\frac{3}{2}}}{27\cdot E\cdot I\cdot(a+b)}$$

$$\mathbf{\Delta AtZeroSlope1}(\mathbf{p},\mathbf{a},\mathbf{b},\mathbf{E},\mathbf{I}) \rightarrow -\frac{\sqrt{3}\cdot \mathbf{a}^{\frac{3}{2}}\cdot \mathbf{b}\cdot \mathbf{p}\cdot(\mathbf{a}+2\cdot \mathbf{b})^{\frac{3}{2}}}{27\cdot \mathbf{E}\cdot \mathbf{I}\cdot(\mathbf{a}+\mathbf{b})}$$

$$\mathbf{\Delta AtZeroSlope2}(\mathbf{p},\mathbf{a},\mathbf{b},\mathbf{E},\mathbf{I}) := \Delta 2\left(b-\frac{\sqrt{3}\cdot\sqrt{b\cdot(2\cdot a+b)}}{3},p,a,b,E,I\right) \xrightarrow{\textit{simplify}} \ldots$$

$$\mathbf{\Delta AtZeroSlope2}(\mathbf{p},\mathbf{a},\mathbf{b},\mathbf{E},\mathbf{I}) \rightarrow \frac{\mathbf{a}\cdot\mathbf{p}\cdot\left(18\cdot\mathbf{a}\cdot\mathbf{b}^2+\sqrt{3}\cdot(\mathbf{b}^2+2\cdot\mathbf{a}\cdot\mathbf{b})^{\frac{3}{2}}-3\cdot\sqrt{3}\cdot\mathbf{b}^2\cdot\sqrt{\mathbf{b}^2+2\cdot\mathbf{a}\cdot\mathbf{b}}-6\cdot\sqrt{3}\cdot\mathbf{a}\cdot\mathbf{b}\cdot\sqrt{\mathbf{b}^2+2\cdot\mathbf{a}\cdot\mathbf{b}}\right)}{54\cdot\mathbf{E}\cdot\mathbf{I}\cdot(\mathbf{a}+\mathbf{b})}$$

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$$\Delta Max(p,a,b,E,I) := \begin{cases} \Delta 1(a,p,a,b,E,I) + \Delta AtZeroSlope2(p,a,b,E,I) & \text{if } a<b \\ \Delta AtZeroSlope1(p,a,b,E,I) & \text{else} \end{cases}$$

$\Delta Max(P,A,B,E_1,I_1) = -0.0206\ in$

$$\Delta MaxLocation(p,a,b,E,I) := \begin{cases} a+b-\dfrac{\sqrt{3}\cdot\sqrt{b\cdot(2\cdot a+b)}}{3} & \text{if } a<b \\ \dfrac{\sqrt{3}\cdot\sqrt{a}\cdot\sqrt{a+2\cdot b}}{3} & \text{else} \end{cases}$$

$\Delta MaxLocation(P,A,B,E_1,I_1) = 9.18\ ft$

Check

$\Delta 3(Distance,P,A,B,E_1,I_1) = -0.0206\ in$

$Distance = 9.18\ ft$

Engineering Example A1
Beam with Point Load

Now, find the minimum value of I1 to limit the deflection to a specified amount relative to the length.

Deflection Criteria (DC) Max Deflection = L/DC $DC := 240$

$I_1 = 428.00\ in^4$ $\quad \dfrac{Length}{DC} = 1.00\ in$

Solver Constraints Values

$$-\Delta 3\left(Distance, P, A, B, E_1, I_1\right) = \frac{Length}{DC}$$

$$I_{min} := \text{Find}\left(I_1\right) = 8.798\ in^4$$

Check

$$\Delta 3\left(\text{Distance}, \mathbf{P}, \mathbf{A}, \mathbf{B}, \mathbf{E}_1, \mathbf{I}_{min}\right) = -1.00\ in$$

Combine Point Load with Uniform Load

Combine the above functions to calculate the point load with the functions derived in Engineering Example 13.1 which calculated the functions for a uniform load.

$$\text{ShearUniform}(x, w, L) := \frac{L \cdot w}{2} - w \cdot x$$

$$\text{MomentUniform}(x, w, L) := \frac{w \cdot x \cdot (L - x)}{2}$$

$$\text{SlopeUniform}(x, w, L, E, I) := \frac{-w \cdot \left(L^3 - 6 \cdot L \cdot x^2 + 4 \cdot x^3\right)}{24 \cdot E \cdot I}$$

$$\Delta\text{Uniform}(x, w, L, E, I) := -\frac{w \cdot x \cdot \left(L^3 - 2 \cdot L \cdot x^2 + x^3\right)}{24 \cdot E \cdot I}$$

$$\text{CombShear}(x, w, p, a, b) := \text{Shear}(x, p, a, b) + \text{ShearUniform}(x, w, a + b)$$

$$\text{CombMoment}(x, w, p, a, b) := \text{Moment}(x, p, a, b) + \text{MomentUniform}(x, w, a + b)$$

$$\text{CombSlope}(x, w, p, a, b, E, I) := \text{Slope}(x, p, a, b, E, I) + \text{SlopeUniform}(x, w, a + b, E, I)$$

$$\text{Comb}\Delta(x, w, p, a, b, E, I) := \Delta 3(x, p, a, b, E, I) + \Delta\text{Uniform}(x, w, a + b, E, I)$$

$W := 120\ plf$

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The maximum moment occurs at the location of zero shear. Solve for the location of zero shear.

Solve Block

Guess Values

$x := 10\ ft$

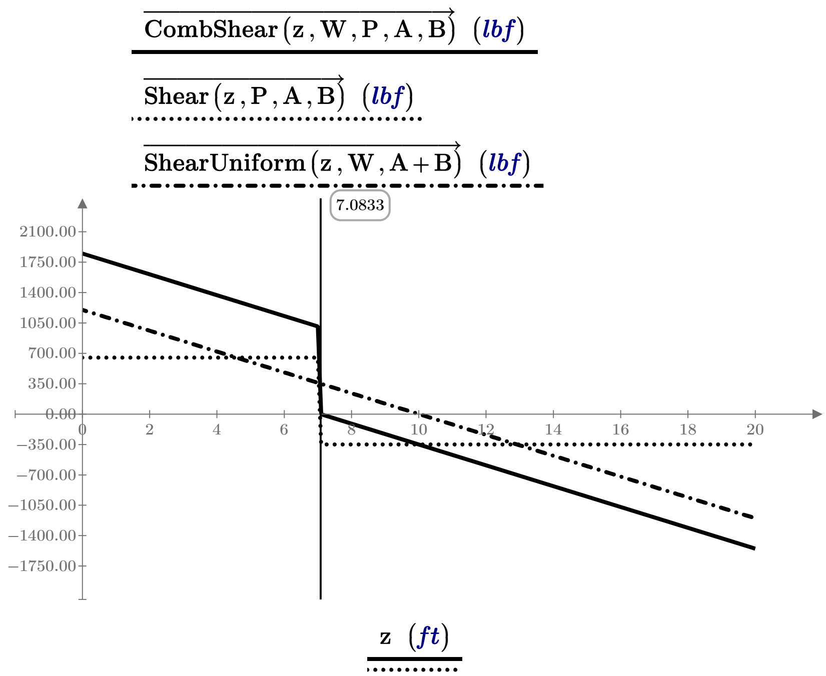
Constraints

$0 = CombShear(x, W, P, A, B)$

Solver

$MDist := Find(x) = 7.08\ ft$

$MDist = 7.08\ ft$



Engineering Example A1
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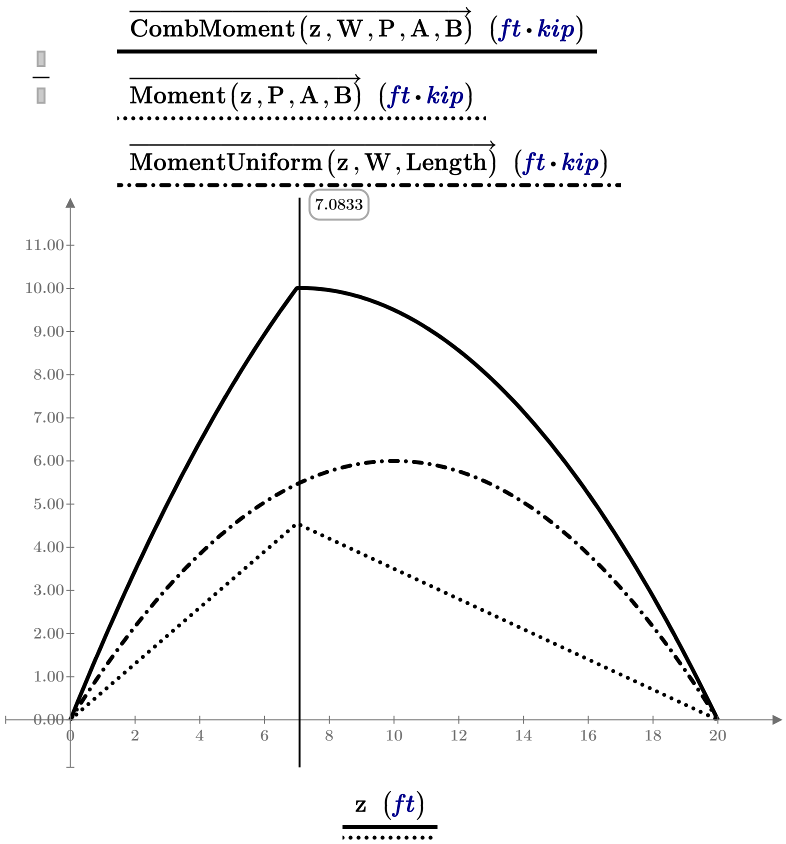
Plot the moment curve and calculate the maximum moment at the location of zero shear.



$$\mathrm{MaxMoment} := \underline{max}\left(\overrightarrow{\mathrm{CombMoment}(z, W, P, A, B)}\right) = 10.01\ ft \cdot kip$$

Calculate the moment at the location of zero shear.

$$\mathrm{MDist} = 7.08\ ft$$

$$\mathrm{CombMoment}(\mathrm{MDist}, W, P, A, B) = 10.01\ ft \cdot kip$$

Engineering Example A1
Beam with Point Load

The point of maximum momement occurs where the slope is zero. Solve for the location of zero slope.

Guess Values

$x := 10\ ft$

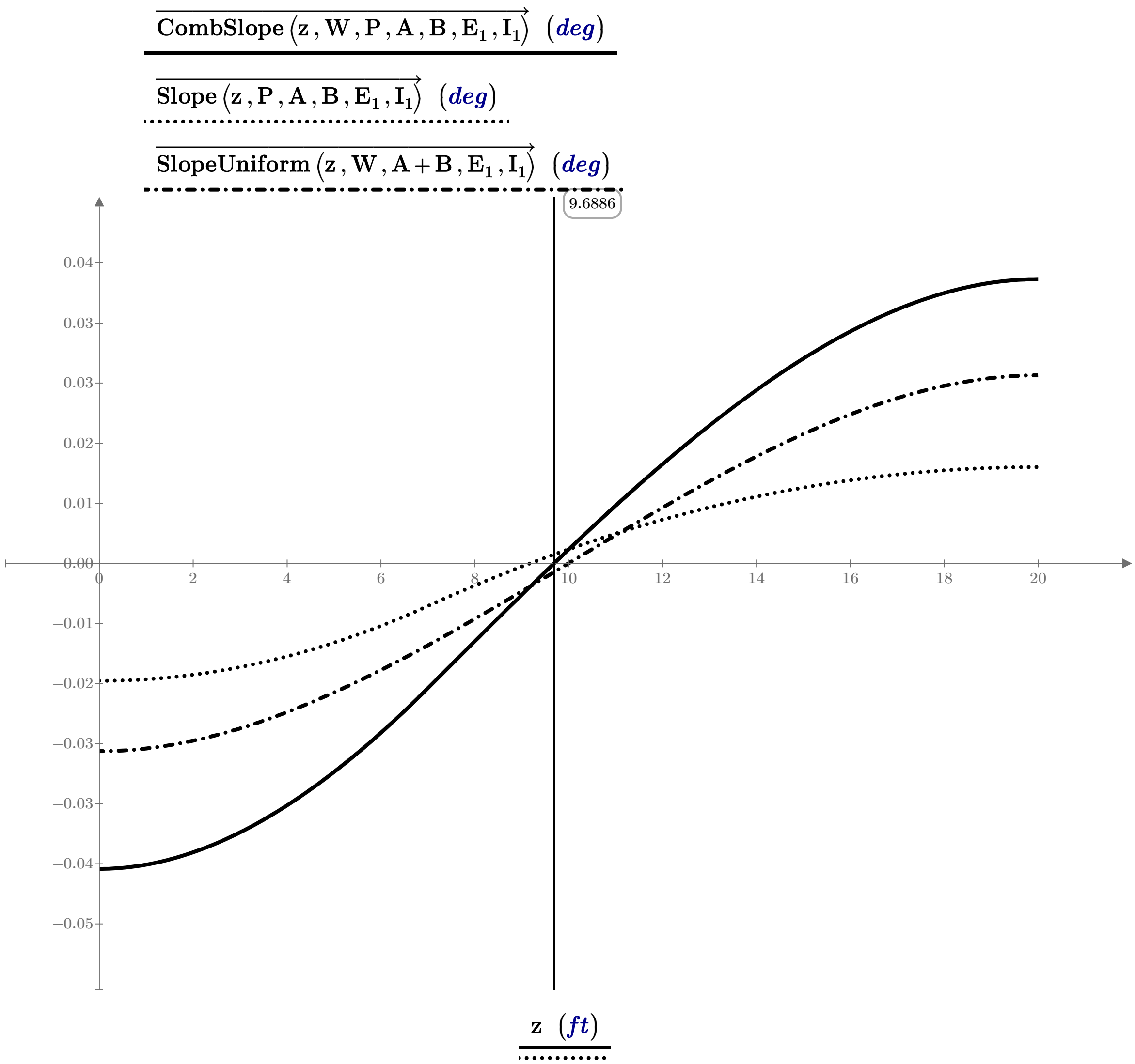
Constraints

$0 = CombSlope\left(x, W, P, A, B, E_1, I_1\right)$

Solver

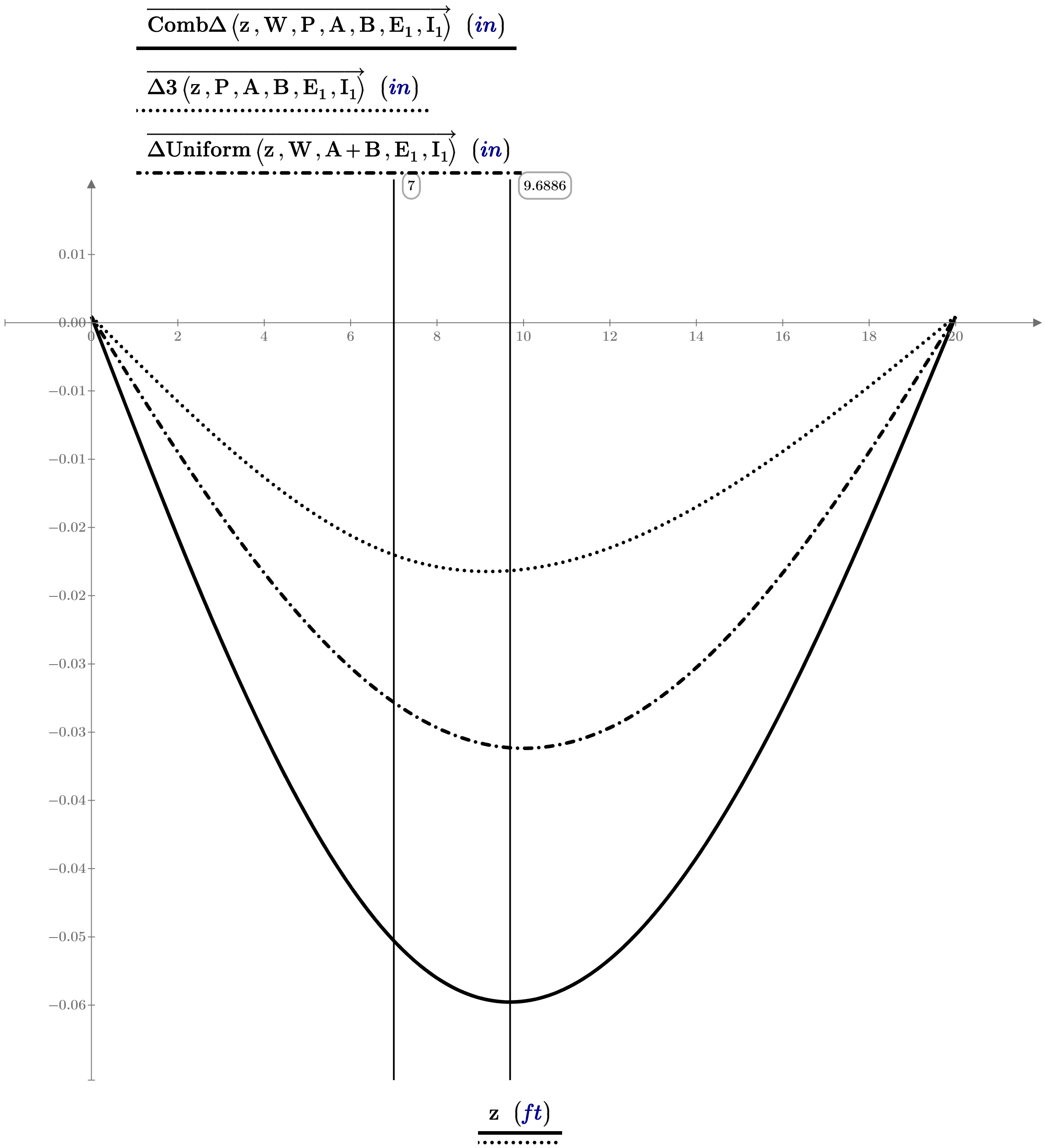
$\Delta Dist := Find(x) = 9.69\ ft$

$\Delta Dist = 9.6886\ ft$



Engineering Example A1
Beam with Point Load

Plot the deflection curve and calculate the maximum deflection at the location of zero slope.



The point of maximum momement occurs where the shear is zero.

$$\text{Comb}\Delta(\Delta\text{Dist}, W, P, A, B, E_1, I_1) = -0.06\ in$$